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# ITERATED LOGARITHMS OF ENTIRE FUNCTIONS

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ABSTRACT
We characterize those sequences $\{f_n\}$ of entire functions satisfying $f_n = \exp(f_{n+1})$ for all $n$.

## I. Introduction and preliminary lemmas

It is a fundamental result in complex analysis that a non-vanishing entire function $f$ admits the representation $f = e^g$ for some entire function $g$. Indeed, in this case, the functions

$$z \to g(z) + 2n\pi i, \qquad n \in \mathbf{Z}$$

give an infinite set of such logarithms of $f$. If one of these logarithms, which we denote by $f_1$, is itself nonvanishing, then we can write

$$f = e^{f_1};\ f_1 = e^{f_2};\ f_2 \text{ entire.}$$

One can conceive of this process continuing indefinitely; if so, we should arrive at a sequence of entire functions $\{f_n\}$, $n = 0, 1, 2 \ldots$, with $f_0 = f$ and with

$$f_n = e^{f_{n+1}}, \qquad n = 0, 1, 2 \cdots. \tag{1}$$

Professor K. Kunen (private communication) has asked whether such sequences exist in any but the trivial case $f_n \equiv$ constant. Our purpose in this note is to characterize all such sequences $\{f_n\}$. Indeed, we shall prove the following result.

THEOREM. *For each sequence of complex numbers* $\{a_n\}_{n=0}^{\infty}$ *with* $a_n = e^{a_{n+1}}$, $n = 1, 2, \ldots$, *and for each nonconstant entire function* $\phi$ *with* $\phi(0) = 0$, *there exists a unique sequence of entire functions* $\{f_n\}_{n=0}^{\infty}$ *satisfying*

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(i) $f_n = e^{f_{n+1}}$ *for all* $n$,

(ii) $f_n(0) = a_n$ *for all* $n$,

(iii) $\lim_{n\to\infty} \frac{f_n''}{f_n'} = \frac{\phi''}{\phi'}$ *uniformly on compact subsets of the set* $\{z \in \mathbf{C}: \phi'(z) \neq 0\}$,

(iv) $\lim_{z\to 0} \frac{f_0'(z)}{\phi'(z)} = 1$.

*Every sequence of nonconstant entire functions* $\{f_n\}$ *satisfying* (i) *may be obtained in this way, for some unique* $\phi$ *and* $\{a_n\}$.

The proof of the Theorem is somewhat long, and will be accomplished via several lemmas.

At the outset, we make some general remarks. We note first that if our $f_0$ is nonconstant, then there is at most one possibility for $f_1$, since Picard's Theorem shows that at most one of the logarithms of $f_0$ is nonvanishing. Inductively, we see that $f_0$ determines the entire sequence $\{f_n\}$.

We also note here that if we can construct one nonconstant sequence $f_n$ as in (1), then we immediately obtain a multitude of other examples; indeed, for any entire function $\phi$, $\{f_n \circ \phi\}$ is such an example.

The next observation is fundamental. It follows from (1) that $f_0' = f_0 f_1'$, and by induction that

$$f_0' = f_0 f_1 f_2 \cdots f_n f_{n+1}', \qquad n = 0, 1, 2, \cdots. \tag{2}$$

Thus if the sequence of *numbers* $\{a_n\}$ is given, with $a_n = f_n(0)$, and hence, with $a_n = e^{a_{n+1}}$, for $n = 0, 1, 2, \cdots$, then $f_0'(0)$ determines all of the numbers $f_n'(0)$; indeed

$$f_n'(0) = \left(\prod_{k=0}^{n-1} a_k\right)^{-1} f_0'(0). \tag{3}$$

In the sequel we shall need estimates on the behaviour of $f_n'$ for large $n$. This leads naturally to the study of products like the one appearing in (3). Indeed, we shall prove the following lemma.

LEMMA 1. *Let* $\{a_n\}$ *be a sequence of complex numbers satisfying* $a_n = e^{a_{n+1}}$, $n = 0, 1, 2 \cdots$. *Then for all nonnegative integers* $N$, *we have*

$$\left| \prod_{n=0}^{N} a_n \right| \geqq C(1.01)^{N+1},$$

*where* $C$ *is a positive constant depending on* $\{a_n\}$, *but not on* $N$.

In view of (3), the implication of lemma 1 on the sequence in (1) is that for all

$z \in \mathbf{C}$, $f_n'(z) \to 0$ as an exponential function of $n$. The lemma itself will emerge as a corollary to a sequence of preliminary lemmas.

DEFINITION. For $z \in \mathbf{C}$, define $e^{(0)}(z) = z$, and for $n = 1, 2, \cdots$, define recursively $e^{(n)}(z) = \exp(e^{(n-1)}(z))$.

LEMMA 2. *Let $z \in \mathbf{C}$. Then for $n = 1, 2, \cdots$,*

$$|\operatorname{Im} e^{(n)}(z)| \leqq \left| \prod_{k=1}^{n} e^{(k)}(z) \right| \quad |\operatorname{Im} z|.$$

PROOF. Let $z = x + iy$ with $x$ and $y$ real. Then

$$|\operatorname{Im} e^z| = |e^x \sin y| \leqq |e^x y| = |e^z \operatorname{Im} z|,$$

which proves the result for $n = 1$.

In general, since $e^{(k)}(z) = \exp[e^{(k-1)}(z)]$, the case $n = 1$ implies

$$\begin{aligned} |\operatorname{Im} e^{(n)}(z)| &\leqq |e^{(n)}(z) \operatorname{Im} e^{(n-1)}(z)| \\ &\leqq |e^{(n)}(z)\, e^{(n-1)}(z) \operatorname{Im} e^{(n-2)}(z)| \\ &\leqq \cdots \leqq \left| \prod_{k=1}^{n} e^{(k)}(z) \right| \quad |\operatorname{Im} z|. \end{aligned}$$

LEMMA 3. *Let $b_0 \in \mathbf{C}$, $|b_0| < 1.01$, and define a sequence $b_k$, $k = 0, 1, 2 \cdots$, by*

$$b_k = e^{(k)}(b_0).$$

*Let $n$ be the smallest integer such that $|b_{n-1}| \geqq 1.01$ while $|b_n| < 1.01$. (Tacitly we assume that such an $n$ exists.) Then*

$$\prod_{k=0}^{n-1} |b_k| \geqq (1.01)^n. \tag{4}$$

PROOF. We begin with some explanatory remarks which will prove useful in the sequel. While it may happen that $|b_1| < 1.01$, we have always that $|b_2| \geqq 1.01$. Indeed

$$|b_2| = |e^{b_1}| = \exp[e^{\operatorname{Re} b_0} \cos \operatorname{Im} b_0] \geqq \exp[e^{-1.01} \cos 1.01] > 1.01.$$

It follows, in particular, that the number $n$ in the statement of the theorem always satisfies $n > 2$, and may be alternatively defined as the smallest integer greater than one such that $|b_n| < 1.01$.

Turning to the proof itself, we denote by $l$ the smallest integer with $1 \leqq l \leqq n - 2$ and with $|\operatorname{Im} b_l| \geqq 1.01$. That such an $l$ exists follows from the chain of implications

$$|b_{n-1}| \geqq 1.01, \qquad |b_n| < 1.01 \Rightarrow |\arg b_{n-1}| > 1.01 \Rightarrow |\operatorname{Im} b_{n-2}| > 1.01.$$

By the choice of $l$, $|\operatorname{Im} b_k| < 1.01$ for $k = 0, 1 \cdots l-1$. Since $1.01 < \pi/3$, we conclude that for $k = 1, 2, 3 \cdots l$,

$$\operatorname{Re} b_k > \tfrac{1}{2}|b_k|.$$

Thus, for such $k$,

$$|b_{k+1}| = e^{\operatorname{Re} b_k} > e^{\frac{1}{2}|b_k|}.$$

By a simple induction, we obtain that

$$|b_{l+1}| > \exp(\tfrac{1}{2}\exp(\tfrac{1}{2}\exp\cdots(\tfrac{1}{2}\exp(\operatorname{Re} b_0)))\cdots),$$

where there are a total of $(l+1)$ exp's in the latter expression.

It is easily seen that regardless of the value of $\operatorname{Re} b_0$ (between $-1.01$ and $1.01$), the fact that $l \geqq 1$, together with the above, yields that

$$|b_{l+1}| \geqq (1.01)^{l+1}.$$

By Lemma 2,

$$1.01 \leqq |\operatorname{Im} b_l| \leqq \left|\prod_{k=1}^{l} b_k\right| \quad |\operatorname{Im} b_0| \leqq \left|\prod_{k=0}^{l} b_k\right|$$

whence

$$(1.01)^{l+2} \leqq \left|\prod_{k=0}^{l+1} b_k\right|.$$

Since

$$|b_k| \geqq 1.01 \quad \text{for} \quad l+2 \leqq k \leqq n-1,$$

the proof is complete.

Lemma 4. *Let $\{a_n\}$, $n = 0, 1, 2 \cdots$, be a sequence of complex numbers satisfying $a_n = e^{a_{n+1}}$ for all $n$. Assume that $|a_0| \geqq 1.01$ but that $|e^{a_0}| < 1.01$. Then for all nonnegative integers $N$,*

$$\prod_{n=0}^{N} |a_n| \geqq (1.01)^{N+1}. \tag{5}$$

Proof. The proof is by induction on $N$, the result being true for $N = 0$ by hypothesis. Suppose that (5) has been verified up to $N-1$. If $|a_N| \geqq 1.01$, we are finished. If not, let $k$ be the largest nonnegative integer less than $N$ such that

$|a_k| \geqq 1.01$ while $|e^{a_k}| < 1.01$. Our assumptions concerning $a_0$ show that such a $k$ exists. On the other hand, the induction hypothesis implies that

$$\prod_{n=0}^{k-1} |a_n| \geqq (1.01)^k.$$

Meanwhile, Lemma 3 implies that

$$\prod_{n=k}^{n=N} |a_n| \geqq (1.01)^{N-k+1}.$$

Combining the above product inequalities, we complete the induction step.

A moment's reflection shows that Lemma 1, stated earlier, follows from Lemma 4, and hence, by the remarks following the statement of Lemma 1, we have succeeded in obtaining information about the derivatives of our original functions $\{f_n\}$. The following related result has the advantage of a conceptual, function-theoretic proof.

LEMMA 5. *Suppose that* $\{f_n\}$, $n = 0, 1, \cdots$, *is a sequence of entire functions satisfying*

$$f_0 = e^{(n)}(f_n) \quad \textit{for all } n.$$

*Then* $\lim_{n\to\infty} f'_n = 0$ *uniformly on compact subsets of* $\mathbf{C}$.

PROOF. Given $\varepsilon > 0$ and $0 < R < \infty$, we must find an integer $n_0$ such that if $n > n_0$, then $|f'_n(z)| < \varepsilon$ for $|z| < R$. To do so, choose $S > R$ such that $4\pi/S < \varepsilon$, and let $M = \sup_{|z|<2S} |f_0(z)|$. Then choose $n_0$ so large that $e^{(n_0-1)}(0) > M$. Now we note that if $\operatorname{Im}(f_n(z)) = 2\pi k i$ for some $z \in \mathbf{C}$, and for some integers $n$ and $k$, with $n \geqq 1$, then $f_{n-1}(z)$ is real and positive, whence $f_0(z) > e^{(n-1)}(0)$. It follows that for $n > n_0$ and for $|z| < 2S$, $\operatorname{Im} f_n(z)$ is never of the form $2\pi k i$. Thus each point $z_0$ with $|z_0| < R$ is the center of a disc of radius $S$ over which the variation of $\operatorname{Im} f_n(z)$ ($n > n_0$) does not exceed $2\pi$. Using the Herglotz formula for $f'_n(z)$ in terms of $\operatorname{Im}\{f_n(z) - f_n(z_0)\}$, we obtain the estimate

$$|f'_n(z_0)| \leqq 4\pi/S < \varepsilon. \qquad \text{Q.E.D.}$$

The following is the key lemma on the behavior of our $f'_n$.

LEMMA 6. *With* $\{f_n\}$ *entire functions satisfying* (1)

$$\sum_{n=0}^{\infty} |f'_n(z)| < \infty \quad \textit{for all} \quad z \in \mathbf{C}, \tag{6}$$

*and the convergence is uniform on every compact subset of* $\mathbf{C}$.

PROOF. The pointwise convergence in (6) is an immediate consequence of formula (2) together with Lemma 1. Since uniform convergence on compact subsets is equivalent to uniform convergence in a neighborhood of each point, we shall be content to prove the latter. So we choose $z_0 \in \mathbf{C}$, and we suppose first that for some $k$,

$$|f_k(z_0)| > 1.01 \quad \text{while} \quad |e^{f_k(z_0)}| < 1.01. \tag{7}$$

By the continuity of $f_k$, these relations must remain valid in some relatively compact neighborhood $N$ of $z_0$. From Lemma 4, we conclude that if $z \in N$ and if $l \geqq k$ is an integer, then

$$\prod_{n=k}^{l} |f_n(z)| \geqq (1.01)^{l-k+1}.$$

Thus, by formula (2),

$$|f'_l(z)| \leqq c(1.01)^{k-l}, \qquad z \in N,\ l \geqq k,$$

where $c = \sup_N |f'_k|$. It follows immediately that $\Sigma_{n=0}^{\infty} |f'_n(z)|$ converges uniformly in $N$.

It remains to consider the case where (7) holds for no $k$. In this case one sees that

$$|f_n(z_0)| \geqq 1.01 \quad \text{for} \quad n \geqq 2.$$

It then follows easily from Lemma 5 that the inequality

$$|f_n(z)| > 1.005, \qquad n \geqq 2,$$

persists for all $z$ in a neighborhood $N$ of $z_0$. Using formula (2), we obtain uniform convergence on $N$ of the sum in (6).

LEMMA 7. *With $\{f_n\}$ as in Lemma 6, $\lim_{n\to\infty} f''_n/f'_n$ exists uniformly on every compact subset of* $\mathbf{C}$ *which is disjoint from the zero set of $f'_0$, and thus this limit defines a meromorphic function on all of* $\mathbf{C}$.

PROOF. First we remark that by virtue of formula (2), the zeros of $f'_0$ are identical with the zeros of $f'_n$ for all $n$. It also follows, by logarithmic differentiation of that formula, that

$$\frac{f''_0}{f'_0} = \frac{f'_0}{f_0} + \frac{f'_1}{f_1} + \cdots + \frac{f'_{n-1}}{f_{n-1}} + \frac{f''_n}{f'_n} = f'_1 + f'_2 + \cdots + f'_n + \frac{f''_n}{f'_n} \qquad (n = 1, 2 \cdots). \tag{8}$$

Thus $\lim_{n\to\infty} f''_n/f'_n = f''_0/f'_0 - \Sigma_{n=1}^{\infty} f'_i$, and the latter sum converges uniformly on compact sets by Lemma 6.

The following is our basic existence result.

LEMMA 8. *For every sequence* $\{a_n\}_{n=0}^{\infty}$ *of complex numbers satisfying* $a_n = e^{a_{n+1}}$ *for all n, there exists a unique sequence of entire functions* $\{f_n\}_{n=0}^{\infty}$ *such that* $f_n = e^{f_{n+1}}$ *and* $f_n(0) = a_n$ *for all n,* $f_0'(0) = 1$, *and* $\lim_{n\to\infty} f_n''/f_n' = 0$ *uniformly on compact subsets of* **C**.

PROOF. We shall construct our sequence $\{f_n\}$ by a familiar technique from differential equations. Assuming that the sequence indeed exists, we shall determine from our hypotheses what must be the successive derivatives of each $f_n$ at the origin, thus, incidentally, proving uniqueness. Then we shall construct formal power series for the $f_n$, using these hypothetical derivatives. Finally we shall show that the power series thus formed represent actual entire functions with all of the desired properties.

Slightly abusing notation, let us denote by $f_n^{(k)}$ the $k$th derivative at the origin of our (as yet hypothetical) function $f_n$. Now the values

$$f_n(0) = a_n$$

are given, and the values

$$f_n^{(1)} = \left\{\prod_{p=0}^{n-1} a_p\right\}^{-1} \tag{9}$$

can be read off formula (2) and the hypothesis that $f_0'(0) = 1$. To calculate the higher derivatives $f_n^{(k)}$, we note that formula (8) together with the condition $\lim_{n\to\infty} f_n''/f_n' = 0$ yields

$$f_n^{(2)} = \sum_{m=n+1}^{\infty} f_n^{(1)} f_m^{(1)}, \qquad n = 0, 1, 2 \cdots. \tag{10}$$

Formula (9) and Lemma 1 show that the above sums are absolutely convergent, and thus, that the numbers $f_n^{(2)}$ are well defined. Most importantly, (10), viewed as a functional identity involving a uniformly convergent sum, can be repeatedly differentiated, thus yielding a family of identities expressing each derivative $f_n^{(k)}$ ($n \geqq 0$, $k \geqq 2$) in terms of lower order derivatives. Such a family of identities permits a recursive determination of all of the numbers $f_n^{(k)}$, which completes the proof of our uniqueness assertion.

To show that the $f_n^{(k)}$ obtained by the above process are indeed the successive derivatives of some entire functions, we must estimate their size. To that end, we define the numbers

$$b_n = |f_n^{(1)}|$$

$$n = 0, 1, 2 \cdots.$$

$$r_n = \sum_{m=n+1}^{\infty} |f_m^{(1)}|$$

By formula (9) and Lemma 1, the $r_n$ are finite and indeed $\lim_{n\to\infty} r_n = 0$.

We claim that

$$\text{(i)} \quad |f_n^{(k)}| \leqq (k-1)!\, b_n r_n^{k-1},$$

$$\text{(ii)} \quad \sum_{m=n+1}^{\infty} |f_m^{(k)}| \leqq (k-1)!\, r_n^k, \qquad \begin{matrix} n = 0, 1 \cdots \\ k = 1, 2 \cdots \end{matrix}$$

(i) and (ii) are proved by induction on $k$, the case $k = 1$ holding for all $n$ by definition of $b_n$ and $r_n$. Assume that (i) and (ii) have been proved for all $n$ when $k = 1, 2 \cdots, s + 1$, for some $s \geqq 0$. We note that according to formula (10) and the Leibniz rule of differentiation, we have for each $n$

$$f_n^{(s+2)} = \sum_{m=n+1}^{\infty} \sum_{q=0}^{s} \binom{s}{q} f_m^{(q+1)} f_n^{(s-q+1)}.$$

Using the induction hypothesis, we obtain that

$$\begin{aligned} |f_n^{(s+2)}| &\leqq \sum_{q=0}^{s} \binom{s}{q} |f_n^{(s-q+1)}| \sum_{m=n+1}^{\infty} |f_m^{(q+1)}| \\ &\leqq \sum_{q=0}^{s} \frac{s!}{q!\,(s-q)!} (s-q)!\, b_n r_n^{s-q} (q!\, r_n^{q+1}) \\ &= \sum_{q=0}^{s} s!\, b_n r_n^{s+1} = (s+1)!\, b_n r_n^{s+1}, \end{aligned}$$

which verifies (i) with $k = s + 2$.

Similarly,

$$\begin{aligned} \sum_{m=n+1}^{\infty} |f_m^{(s+2)}| &\leqq \sum_{m=n+1}^{\infty} \sum_{q=0}^{s} \binom{s}{q} |f_m^{(s-q+1)}| \sum_{t=m+1}^{\infty} |f_t^{(q+1)}| \\ &\leqq \sum_{q=0}^{s} \binom{s}{q} \sum_{m=n+1}^{\infty} |f_m^{(s-q+1)}| \sum_{t=n+1}^{\infty} |f_t^{(q+1)}| \\ &\leqq \sum_{q=0}^{s} \frac{s!}{q!\,(s-q)!} (s-q)!\, r_n^{s-q+1} (q!) r_n^{q+1} \\ &= (s+1)!\, r_n^{s+2}, \end{aligned}$$

which verifies (ii) with $k = s + 2$.

It follows immediately from (i) that the power series $a_n + \sum_{k=1}^{\infty} f_n^{(k)} z^k / k!$ has radius of convergence at least $1/r_n$. We denote by $f_n$ the analytic function defined by this series, and we note that from (ii) it follows easily that $\sum_{n=0}^{\infty} f'_n$ converges uniformly on compact subsets of the disc $\{|z| < 1/r_0\}$. In this disc, we can therefore consider the functions

$$f''_n - \sum_{p=n+1}^{\infty} f'_n f'_p, \qquad n = 0, 1, 2, \cdots.$$

From the way our $f_n$ were defined, it is clear that these functions and all of their derivatives vanish at the origin. The conclusion is that for $|z| < 1/r_0$

$$f''_n(z)/f'_n(z) = \sum_{p=n+1}^{\infty} f'_p(z), \qquad n = 0, 1, 2, \cdots. \tag{11}$$

In particular, for all $n$, we have in $\{|z| < 1/r_0\}$

$$f''_n/f'_n - f''_{n+1}/f'_{n+1} = f'_{n+1}.$$

From this formula and from our choice of values at zero for $f_n$ and $f'_n$ $(n = 0, 1, 2, \cdots)$ it follows easily that

$$f_n(z) = e^{f_{n+1}(z)}, \qquad n = 0, 1, 2, \cdots; \ |z| < 1/r_0. \tag{12}$$

Now we recall that $f_n(z)$ is actually analytic in $|z| < 1/r_n$. Since $\lim_{n\to\infty} r_n = 0$, formula (12) may be employed to give an analytic continuation of each $f_n$ to an entire function. Meanwhile, we have trivially that $f_n(0) = a_n$ for all $n$ and that $f'_0(0) = 1$; so it remains only to prove that

$$\lim_{n\to\infty} f''_n/f'_n = 0 \tag{13}$$

uniformly on compact subsets of **C**. However, Lemma 6 now shows that the sums on the right hand side of formula (11) converge uniformly on compact subsets of **C**, and so by analytic continuation, (11) is an identity of entire functions for each $n$. (13) follows immediately, completing the proof of Lemma 8.

## II. The main result and its ramifications

We are finally prepared to prove our main result, whose statement we repeat for convenience.

THEOREM. *For each sequence of complex numbers* $\{a_n\}_{n=0}^{\infty}$ *with* $a_n = e^{a_{n+1}}$, $n = 1, 2, \cdots$, *and for each nonconstant entire function* $\phi$ *with* $\phi(0) = 0$, *there exists a unique sequence of entire functions* $\{g_n\}_{n=0}^{\infty}$ *satisfying*

(i) $g_n = e^{g_{n+1}}$ *for all* $n$,

(ii) $g_n(0) = a_n$ *for all* $n$,

(iii) $\lim_{n\to\infty} g''_n/g'_n = \phi''/\phi'$ *uniformly on compact subsets of the set* $\{z \in \mathbf{C}: \phi'(z) \neq 0\}$,

(iv) $\lim_{z\to 0} g'_0(z)/\phi'(z) = 1$.

*Every sequence of nonconstant entire functions* $\{g_n\}$ *satisfying* (i) *may be obtained in this way, for some unique* $\phi$ *and* $\{a_n\}$.

Proof. The theorem makes essentially three assertions, concerning existence, uniqueness, and completeness.

For the existence proof, we merely note that if $\{f_n\}$ is the sequence constructed in Lemma 8 corresponding to the numbers $\{a_n\}$, then the sequence $\{g_n\} = \{f_n \circ \phi\}$ has all of the properties (i)→(iv). In particular,

$$\lim_{n\to\infty} \frac{(f_n \circ \phi)''}{(f_n \circ \phi)'} = \lim_{n\to\infty} \left\{ \frac{(f''_n \circ \phi)}{(f'_n \circ \phi)} \phi' + \frac{\phi''}{\phi'} \right\} = \frac{\phi''}{\phi'}$$

since $\lim_{n\to\infty} f''_n/f'_n = 0$. This proves (iii), and (iv) follows from the fact that $f'(0) = 1$.

Turning to the uniqueness assertion, we follow the lines of the proof of Lemma 8. Indeed, assuming that the $g_n$ exist, we show how to recover their power series from the hypotheses of the theorem. We use primarily the formula

$$g''_n = \sum_{m=n+1}^{\infty} g'_n g'_m + g'_n \frac{\phi''}{\phi'}, \qquad n = 0, 1, \cdots, \tag{14}$$

which is derived from formula (8). If $\phi'(0) \neq 0$, then $\phi''/\phi'$ is analytic near 0, and the process of recovering the $g_n$ is essentially identical to the process used in the proof of Lemma 8. On the other hand, if $\phi'(0) = \phi''(0) = \cdots = \phi^{(k)}(0) = 0$, it follows from (iv) that all of these derivatives vanish likewise for $g_0$, and by repeated differentiation of our formula (2), we see that the same holds for each $g_n$. Moreover, if $\phi^{(k+1)}(0) = \alpha \neq 0$, then (iv) yields that $g_0^{(k+1)}(0) = \alpha$, and a differentiated version of formula (2) shows that

$$g_n^{(k+1)}(0) = \alpha \left\{ \prod_{s=0}^{n-1} a_s \right\}^{-1}.$$

Meanwhile, in this case, $\phi''/\phi' = k/z + F$, where $F$ is analytic in a neighborhood of zero. Thus we have the relations

$$g''_n - k(g'_n/z) = \sum_{m=n+1}^{\infty} g'_n g'_m + g'_n F. \tag{14'}$$

But a check of power series shows that since $g'_n(0) = 0$, the $s$th derivative at zero of the function $g'_n/z$ is just $g_n^{(s+2)}(0)/(s+1)$. Thus differentiation of formula (14′) yields the equality

$$g_n^{(s+2)}(0)\left[1-\frac{k}{s+1}\right] = \text{expression involving only } F \text{ and derivatives of the } g_m \text{ up to order } s+1.$$

All of the numbers $g_m^{(s+2)}(0)$ are already known up to and including $s = k-1$, and so the above formula permits a recursive determination of the remaining derivatives, proving our uniqueness assertion.

It remains to prove the completeness assertion of the theorem; namely, that given entire functions $\{g_n\}$ satisfying (i), we can produce an appropriate (and unique) sequence $\{a_n\}$ and function $\phi$ to fit the paradigm of our theorem. Of course the $a_n$ are chosen as $g_n(0)$. To find $\phi$, we recall from the proof of Lemma 7 that

$$\lim_{n\to\infty} g''_n/g'_n = g''_0/g'_0 - \sum_{n=1}^{\infty} g'_n = \frac{g''_0}{g'_0} - G$$

where $G$ is a well-defined entire function. We require an entire function $\phi$ satisfying

$$\frac{\phi''}{\phi'} = \lim_{n\to\infty}\frac{g''_n}{g'_n} = \frac{g''_0}{g'_0} - G.$$

This equation may be integrated; indeed, we obtain the general solution

$$\phi(z) = c_1 + c_2\int_0^z g'_0(w)\exp\left\{-\int_0^w G(\xi)d\xi\right\}dw, \tag{15}$$

and it follows easily that $c_1$ and $c_2$ may always be chosen, uniquely, so as to secure condition (iv) and the requirement $\phi(0) = 0$. In fact, $c_1 = 0$ and $c_2 = 1$. This completes the proof of the theorem.

We remark that the integrations indicated in formula (15) can be carried out in a semi-explicit manner to obtain the rather striking result

$$\phi(z) = c_1 + c_2\lim_{n\to\infty}\frac{g_n(z)-g_n(\alpha)}{g_n^{(k)}(\alpha)}, \tag{16}$$

where $\alpha \in \mathbf{C}$ is arbitrary, $g_n^{(k)}(\alpha)$ is the first non-vanishing derivative of $g_n$ at $\alpha$, and $c_1$ and $c_2$ are constants depending on $\alpha$.

It is not difficult to translate our theorem to the case where the $\{g_n\}$ and $\phi$ are defined in an arbitrary simply connected domain in $\mathbf{C}$. Here one shows that the only possibilities for $\{g_n\}$ are sequences of the form $\{f_n \circ \phi\}$, where the sequence $\{f_n\}$ is as in Lemma 8, and this time $\phi$ is an arbitrary function analytic in the given domain. The formula (16) for recovering $\phi$ remains valid in this more general situation. The obstacle to extending our result to multiply connected domains is that our completeness proof depended on the fact that holomorphic entire functions have holomorphic, single valued, integrals. This fact, of course, has no direct analogy in the case of multiply connected domains.

In conclusion, we should like to discuss one interesting special case of our theorem. That is the case where the entire functions $\{g_n\}$ considered have a common value, say $\alpha$, at the origin. This necessitates $e^{\alpha} = \alpha$, an equation which has infinitely many complex solutions. Indeed, the entire function $ze^{-z}$ is zero only for $z = 0$, and hence, by Picard's Theorem, it takes the value 1 infinitely often. It is easily verified that all of these fixed points of the exponential have modulus greater than 1.01, corroborating Lemma 1. It happens that the functions $\{f_n\}$ constructed in Lemma 8 with respect to the constant sequence $\{\alpha\}$ satisfy the interesting functional equation

$$f_n(\alpha z) = e^{f_n(z)}. \tag{17}$$

Indeed, this follows from the uniqueness assertion of Lemma 8. For if $\{f_n(z)\}_{n=0}^{\infty}$ is the sequence thus constructed, then it and the sequence $\{f_n(\alpha z)\}_{n=1}^{\infty}$ satisfy the same "initial conditions", and hence must be identical. It follows by induction that in this case,

$$f_n(z) = f_0(z/\alpha^n).$$

Using the above relations, one easily verifies the conclusions of Lemmas 6 and 7 in our special case.

We remark without proof that the sequences $\{f_n \circ \phi\}$ with $f_n$ as above and $\phi$ entire can be shown to be identical with the sequences $\{g_n\}$ where the $g_n$ are entire, $g_n = e^{g_{n+1}}$, and $\lim_{n\to\infty} g_n(0)$ exists.

Finally, we note that if we set

$$h_n(z) = f_n(\alpha^z)$$

with $f_n$ as in (17), then we obtain solutions to the functional equation

$$h(z+1) = e^{h(z)}.$$

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